Period 1, Jan 8, 2024

$$g'(x)=\frac{2x(x^2-1)-x^2(2x)}{(x^2-1)^2}=\frac{\cancel{2x^3}-2x-\cancel{2x^3}}{(x^2-1)^2}=\frac{-2x}{(x^2-1)^2}$$

7. If $g(x)=\frac{x^2}{x^2-1}$, find

$g'(0)=0$

$g'(-1)=\varnothing$

$g'(1)=\varnothing$

Slope is + or −

(a) the intervals on which g is increasing or decreasing. Justify your response.

increasing $(-\infty,-1)\cup(-1,0)$

+ + + + + + | + + + | − − − | − − − − − on the number line −3, −2, −1, 0, 1, 2, 3

$g'(-2)=\frac{+}{+}=+$

$g'(-\frac{1}{2})=\frac{+}{+}=+$

$g'(\frac{1}{2})=\frac{-}{+}=-$

decreasing $(0,1)\cup(1,\infty)$

$g'(2)=\frac{-}{+}=-$

$g'(x)=0$ or $\varnothing$

(b) the x-values of local maximum and minimum values. Justify your response.

$g(1)=\frac{1^2}{1^2-1}=\frac{1}{0}=\varnothing$

$g(-1)=\frac{(-1)^2}{(-1)^2-1}=\frac{1}{0}=\varnothing$

$g(0)=\frac{0^2}{0^2-1}=\frac{0}{-1}=0$

$g''(x)=+$ or −

$x=1,-1$

$g(1)=\varnothing$

$g(-1)=\varnothing$

(c) the intervals of concavity and inflection points. Justify your response.

$g''(x)=0$ or $\varnothing$

concave up $(-\infty,-1)\cup(1,\infty)$

concave Down $(-1,1)$

+ + + + | − − − − − − | + + + + + + + on the number line −2, −1, 0, 1, 2

$g''(-2)=\frac{6(-2)^2+2}{((-2)^2-1)^3}=\frac{+}{+}=+$

$g''(0)=\frac{+}{-}=-$

$g''(2)=\frac{+}{+}=+$

$g'(x)=\frac{-2x}{(x^2-1)^2}$

$$g''(x)=\frac{-2(x^2-1)^2-(-2x)(2(x^2-1)^1\cdot 2x)}{((x^2-1)^2)^2}=\frac{\cancel{(x^2-1)}\overbrace{[-2(x^2-1)+8x^2]}^{-2x^2+2+8x^2}}{(x^2-1)^{\cancel{4}3}}$$

$$g''(x)=\frac{6x^2+2}{(x^2-1)^3}$$

$g''(x)=0$

$g''(x)=\varnothing$ when $x=1$ or $x=-1$

$((x-1)(x+1))^3=0$ $x=1$ or $x=-1$

$6x^2+2=0$

$6x^2=-2$

$x^2=-\frac{1}{3}$

$\cancel{x=\sqrt{-\frac{1}{3}}}$

5. Use the Second Derivative Test to find the relative extrema of

a) $f(x)=4x+\frac{4}{x}=4x+4x^{-1}$

$f(0)=4(0)+\frac{4}{0}=\varnothing$

extrema $x=1$ or $x=-1$ Max

b) $h(x)=6x-x^2$

$h'(x)=6-2x \Rightarrow h'(3)=6-2\cdot 3=0$

$h''(x)=-2$ concave Down Every where Max

5a

$F'(x) = 4 - 4x^{-2} = 4 - \frac{4}{x^2}$

$F'(x) = 0$ or ϕ $F'(0) = \phi$ $F'(1) = 0$

$F''(x) = 0 + 8x^{-3} = \frac{8}{x^3}$ $F'(-1) = 0$

$h(3) = 6(3) - 3^2 = 18 - 9 = 9$

$F''(0) = \phi$

$F''(-1) = \frac{8}{(-1)^3} = -8 \Rightarrow x = -1$ concave Down Max

$F''(1) = \frac{8}{1} = +$ concave up min

$F(-1) = 4(-1) + \frac{4}{-1} = -4 - 4 = -8 =$ Max

$F(1) = 4(1) + \frac{4}{1} = 4 + 4 = 8$ min

---

6. If $f(t) = 2t^3 + 3t^2 - 36t$, find

(a) the intervals on which f is increasing or decreasing, Justify your response.

+ + + − − − − − − − + + + +

−4 −3 −2 −1 0 1 2 3

$F'(-4) = 6(-4+3)(-4-2) = + \cdot - \cdot - = +$

$F'(0) = 6(0+3)(0-2) = + \cdot + \cdot - = -$

$F'(3) = 6(3+3)(3-2) = + \cdot + \cdot +$

increasing $(-\infty, -3) \cup (2, \infty)$

decreasing $(-3, 2)$

(b) the t-values of local maximum and minimum values. Justify your response.

$F''(-3) = 12(-3) + 6 = -36 + 6 = -$ concave Down Max

$F(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = -54 + 27 + 108 = 81$

$F''(2) = 12(2) + 6 = 24 + 6 = +$ concave uP min

$F(2) = 2(2)^3 + 3(2)^2 - 36(2) = 16 + 12 - 72 = -44$

(c) the intervals of concavity and inflection points. Justify each response.

− − − − − + + + + + + + +

−2 −1 $-\frac{1}{2}$ 0 1 2

$F'(-3) = 0$

$F'(2) = 0$

$F'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2)$

$F''(x) = 12x + 6$

concave down $(-\infty, -\frac{1}{2})$

concave $(-\frac{1}{2}, \infty)$

$F''(-\frac{1}{2}) = 0$

Example 1: Find the tangent line approximation of $f(x) = 1 + \sin x$ at the point (0, 1). Then use a table to compare the y-values of the linear function with those of $f(x)$ on an open interval containing $x = 0$.

$F'(x) = 0 + \cos x$

$F'(0) = 0 + \cos(0) = 0 + 1 = 1$

$m = 1$

Point $(0,1)$

$y = mx + b$

$y = 1x + b$

$1 = 1(0) + b$

$1 = b$

$y = x + 1$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$x \to 0$

$\sin x \approx x$

Example 2

Use local linearization to approximate $\sqrt{16.5}$. Is your answer and overestimate or an underestimate. Justify your reasoning.

$y = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}} = \frac{-1}{4\sqrt{x^3}}$

Slope $= \frac{1}{2\sqrt{16}} = \frac{1}{2 \cdot 4} = \frac{1}{8}$

$\sqrt{16} = 4$

$(16, 4)$ Point close to $\sqrt{16.5}$

$y = mx + b$

$4 = \frac{1}{8}(16) + b$

$4 = 2 + b$

$2 = b$

$y = \frac{1}{8}x + 2$

$y = \frac{1}{8}(16.5) + 2$

$\frac{16.5}{8} + 2 = 2.$

$2.06 + 2 = 4.06$

$\sqrt{16.5} = \underline{4.0620192023}$

$$\begin{array}{r} 2.0625 \\ 8\overline{)16.500} \\ \underline{16} \\ 050 \\ \underline{48} \\ 20 \\ \underline{16} \\ 40 \end{array}$$

$4.062019202 \approx 2.0625$

$F''(16.5) = -$ concave down

over estimate

$3 \to 3.01 \quad 0.01 = dx$

Your Turn: Estimate $(3.01)^3$ without a calculator. Is your answer and overestimate or underestimate? Justify your answer.

$y = x^3 \Rightarrow x=3 \quad y=27$ Point

$\frac{dy}{dx} = 3x^2 \Rightarrow \text{Slope} = 3(3)^2 = 27$

$y = 27x + b$

$27 = 27 \cdot 3 + b$

$27 = 81 + b$

$-54 = b$

$y = 27x - 54$

$y = 27(3.01) - 54$

$y = 81.27 - 54$

$y = 27.27$

$\frac{\text{change in } y}{\text{over change in } x} = \frac{dy}{dx} = 3x^2$

$(0.01) \cdot \frac{dy}{0.01} = 3x^2(0.01)$

$x=3 \quad y=27$

$x=3.01 \quad y=27+0.27=27.27$

$dy = 0.03x^2 = (0.03)9 = 0.27$

$x=3$

The number of gallons, $P(t)$, of a pollutant in a lake changes at the rate $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ gallons per day, where t is measured in days. There are 50 gallons of the pollutant in the lake at time $t = 0$. The lake is considered to be safe when it contains 40 gallons or less of pollutant.

(d) An investigator uses the tangent line approximation to $P(t)$ at $t = 0$ as a model for the amount of pollutant in the lake. At what time t does this model predict that the lake becomes safe?

Point $(0, 50)$

Slope $P'(0) = 1 - 3e^{-0.2\sqrt{0}} = 1 - 3 \cdot e^{0} = 1 - 3 \cdot 1 = -2$

$y = -2x + b$

$50 = -2(0) + b$

$50 = b$

$y = -2x + 50$

$40 > -2x + 50$

$5 < x$

Let f be a function for which $f(2) = 6$ and $f'(2) = -3$.

If the tangent line to a graph of f at 2 is used to approximate a zero of f, then the approximation is

Point $(2, 6)$

Slope $= -3$

$y - 6 = -3(x - 2)$

$y = -3x + 6 + 6$

$y = -3x + 12$

$y = 0$ when $x = 4$

$x \approx 4$

As Seen on MCQ

A linear approximation to $f(x) = x \cdot \sin\left(\frac{\pi x}{2}\right) + x^2$ at $x = 3$ is

$F'(x) = 1 \cdot \sin\frac{\pi x}{2} + x \cdot \left(\cos\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right) + 2x$

$\sin\frac{3\pi}{2} = -1$

$F'(3) = \sin\frac{3\pi}{2} + 3\left(\cos\frac{3\pi}{2}\right)\left(\frac{\pi}{2}\right) + 2(3)$

$\cos\frac{3\pi}{2} = 0$

$F'(3) = -1 + 3 \cdot 0 \cdot \frac{\pi}{2} + 6 = 5 =$ slope

$F(3) = 3 \cdot \sin\frac{3\pi}{2} + 3^2 = 3(-1) + 9 = 6$ point $(3, 6)$

Example 3: Find the derivative of each function in differential form.

a) $y = 2x^3 + 5x^2 - 3x + 1$

$dx \cdot \frac{dy}{dx} = (6x^2 + 10x - 3)\,dx$

$dy = (6x^2 + 10x - 3)\,dx$

Example 5: Inflating a bicycle tire changes its radius from 12 inches to 13 inches. Use differentials to estimate the change in perimeter of the tire.

$Cir = 2\pi r$

Radius starts at 12 inches so $r = \underline{12\ inches}$

Radius goes from $\underbrace{12 \text{ to } 13}$ so $\underbrace{dr = 1}\ inch$

$\frac{dc}{dr} = 2\pi$

$C = 2\pi r$

$C = 2 \cdot \pi \cdot 12 = 24\pi$

Point $(12, 24\pi)$

$(13, 24\pi + 2\pi)$

$(13, 26\pi)$

$dc = 2\pi \cdot dr$

$dc = 2\pi \cdot 1$

$dc = 2\pi$

Change in c

$dx = 0.03$

If $f'(x) = 2xe^{x^2-1} - 3\pi\sin(\pi x)$ and $f(1) = 4$, approximate $f(1.03)$ using a linear approximation.

$F'(1) = 2(1) \cdot e^{1^2-1} - 3\pi \sin\pi$ $\quad (1, 4)$

$2e^0 - 3\pi \cdot 0$

$2 \cdot 1 - 0 = 2$

$\frac{dy}{dx} = 2$

$dy = 2 \cdot dx = 2(0.03)$

$dy = 0.06$

$\frac{dy}{dx} = 2 = \text{Slope}$ $\quad$ Point $(1, 4)$

$y = 2x + b$

$4 = 2(1) + b$

$2 = b$

$y = 2x + 2$

$y = 2(1.03) + 2 = 2.06 + 2 = 4.06$

Point

$(1, 4)$

$dx \downarrow \quad \downarrow dy = 0.06$

$(1.03, 4 + 0.06)$

$(1.03, 4.06)$

Same